CLASS24

PAPER CODE	-
FORM NUMBER	

Genius Seed Program

(ACADEMIC SESSION 2023 – 2024)

National Mathematics Talent Contest 2023

MOCK TEST – 2 (Junior)

Time : 3 Hours

PLEASE READ THE FOLLOWING INSTRUCTIONS CAREFULLY

- Rulers and compasses are allowed.
- Answer all questions. Each question carries 10 marks.
- Elegant and innovative solutions will get extra marks.
- Diagrams and justification should be given wherever necessary.
- Before answering, fill in the FACE SLIP completely.
- Your 'rough work' should be in the answer sheet itself.
- The maximum time allowed is THREE hours.

Name of the Candidate (in Capitals) ____________________

Form Number : ____________________

Centre of Examination (In Capitals) : ____________________

Candidates's Signature :

Invigilator's Signature :

Time : 3 hours **Mathematics : Mock Test -2**

1.

(a) If

$$x = \left(b^{\frac{2015}{2016}} - a^{\frac{2015}{2016}}\right)^{\frac{2016}{2015}}$$

find the value of

$$\sqrt[2015]{x^{2015} + \sqrt[2016]{a^{2015}x^{(2015)^2}}} + \sqrt[2015]{a^{2015} + \sqrt[2016]{x^{2015}a^{(2015)^2}}} - b$$

(b) If

$$N = \sqrt{1 + \frac{1}{1^2} + \frac{1}{2^2}} + \sqrt{1 + \frac{1}{2^2} + \frac{1}{3^2}} + \cdots + \sqrt{1 + \frac{1}{2014^2} + \frac{1}{2015^2}}$$

find $[N]$, the integral part of N.

(a) Let $p = 2015$ and $q = 2016$. We have $q = p + 1$. We have

$$x^{p/q} = b^{p/q} - a^{p/q}$$

Now,

$$\begin{aligned} x^p + \sqrt[q]{a^p x^{p^2}} &= x^p + a^{p/q}x^{p^2/q} \\ &= x^p + \left(b^{p/q} - x^{p/q}\right)x^{p^2/q} \\ &= x^p + b^{p/q}x^{p^2/q} - x^{p/q+p^2/q} \\ &= x^p + b^{p/q}x^{p^2}q - x^{(p/q)(1+p)} \\ &= x^p + b^{p/q}x^{p^2/q} - x^p \text{ since } p+1=q \\ &= b^{p/q}x^{p^2/q} \end{aligned}$$

Similarly,

$$a^p + \sqrt[q]{x^p a^{p^2}} = b^{p/q} a^{p^2/q}$$

Hence the given expression equals

$$\begin{aligned}
&\left(b^{p/q} x^{p^2/q}\right)^{1/p} + \left(b^{p/q} a^{p^2/q}\right)^{1/p} - b \\
&= b^{1/q}\left(x^{p/q} + a^{p/q}\right) - b \\
&= b^{1/q} b^{p/q} - b = b^{(p+1)/q} - b \\
&= b - b = 0
\end{aligned}$$

(b) Let t_n denote the n-th term.

$$t_n = \sqrt{1 + \frac{1}{n^2} + \frac{1}{(n+1)^2}}$$

$$= \sqrt{\left(1 + \frac{1}{n}\right)^2 - \frac{2}{n} + \frac{1}{(n+1)^2}}$$

$$= \sqrt{\left(\frac{n+1}{n}\right)^2 - \frac{2}{n} + \frac{1}{(n+1)^2}}$$

$$= \sqrt{\left(\frac{n+1}{n} - \frac{1}{n+1}\right)^2}$$

$$= \sqrt{\left(1 + \frac{1}{n} - \frac{1}{n+1}\right)^2}$$

$$= 1 + \frac{1}{n} - \frac{1}{n+1}$$

Hence

$$N = \left(1 + \frac{1}{1} - \frac{1}{2}\right) + \left(1 + \frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(1 + \frac{1}{2014} - \frac{1}{2015}\right)$$

$$= 2015 - \frac{1}{2015}$$

Hence $[N] = 2014$.

2.

(a) a, b, c are nonzero real numbers such that $a+b+c=abc$ and $a^2=bc$. Prove that $a^2 \geq 3$.

(b) Find all prime numbers x, y, z such that

$$x(x+y)=z+120$$

Solution

(a) We have $b+c=abc-a=a^3-a$ and $bc=a^2$. Thus b, c are the roots of the equation

$$t^2-(a^3-a)t+a^2=0$$

Since b, c are real, the discriminant of this quadratic ≥ 0. Thus

$$\begin{aligned}&(a^3-a)^2-4a^2 \geq 0\\ \Rightarrow\ &a^2(a^2+1)(a^2-3) \geq 0\\ \Rightarrow\ &a^2 \geq 3\end{aligned}$$

(b) If $z=2$, then we have $x(x+y)=122$. Since x is a prime and $2, 61$ are the only factors of 122, it follows that $x=2$ or $x=61$. But if $x=61$ then $x+y=2$, and y becomes negative. Thus $x=2, y=59, z=2$ is one solution.

If z is odd, $x, x+y$ are both odd and hence y is even. Thus $y=2$. In this case, we have

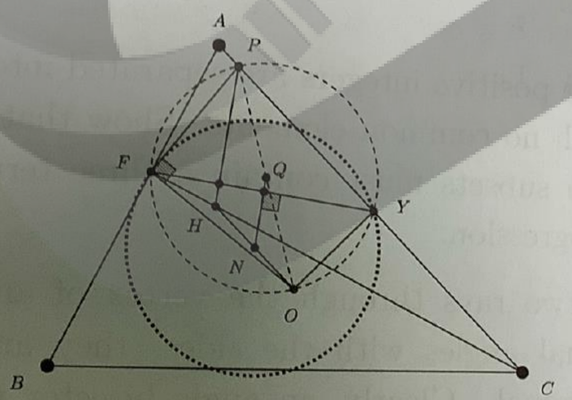
$$x(x+2)=z+122 \Rightarrow (x-10)(x+12)=z$$

Since z is a prime, it follows that $x-10=1$ and $z=23$. Thus $x=11, y=2, z=$ ~~13~~ 23 is the other solution.

3.

Let ABC be an acute angled triangle with $BC > AC$. Let O be the circumcenter and H, the orthocenter of the triangle ABC. F is the foot of the perpendicular from C on AB and the perpendicular to OF at F meets the side CA at P. Show that $\angle FHP = \angle A$.

Solution Let Y be the midpoint of AC. Clearly, $OY \perp AC$. Since $\angle OFP = 90°$, it follows that the quadrilateral $OFPY$ is cyclic. The center of the circumcircle of $OFPY$ is the midpoint Q of OP. Let N be the nine point center of the triangle ABC. Then N is the midpoint of OH and the nine point circle passes through F and Y. Thus the line NQ is the line joining the centers of these two circles and FY is their common chord. Hence $NQ \perp FY$. Also from the triangle OPH, since $OQ = QP, ON = NH$, it follows that $NQ \parallel HP$



and thus $HP \perp FY$. Since $CF \perp AB$, it follows that the angle between HP and CF is equal to the angle between their perpendiculars FY and AB. Thus $\angle FHP = \angle YFA$. But since Y is the midpoint of CA and the perpendicular from Y on AB is parallel to CF, it also bisects AF. Thus $\angle YFA = \angle YAF = \angle A$.

4.

If ${}^{n}C_r = \dfrac{n!}{r!(n-r)!}$, *then prove*

$$\sqrt{C_1} + \sqrt{C_2} + \ldots + \sqrt{C_n} \le 2^{n-1} + \frac{n-1}{2}$$

Ans.6

Consider

$$(\sqrt{C_1} - 1)^2 + (\sqrt{C_2} - 1)^2 + (\sqrt{C_3} - 1)^2 + \ldots + (\sqrt{C_n} - 1)^2 \ge 0$$

$$(C_1 + C_2 + C_3 + \ldots + C_n) - 2(\sqrt{C_1} + \sqrt{C_2} + \ldots + \sqrt{C_n}) + n \ge 0$$

$$2^n - 1 + n \ge 2(\sqrt{C_1} + \sqrt{C_2} \ldots + \sqrt{C_n}) \qquad (\because\ C_0 + C_1 + C_2 + \ldots + C_n = 2^n)$$

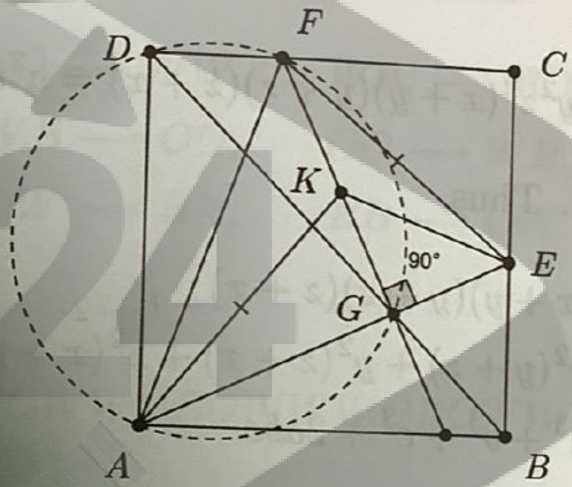
$$(\sqrt{C_1} + \sqrt{C_2} + \ldots + \sqrt{C_n}) \le \frac{2^n + n - 1}{2}$$

$$(\sqrt{C_1} + \sqrt{C_2} + \ldots + \sqrt{C_n}) \le 2^{n-1} + \frac{(n-1)}{2}$$

Hence proved.

5.

● $ABCD$ is a square E and F are points on BC and CD respectively such that AE cuts the diagonal BD at G and FG is perpendicular to AE. K is a point on FG such that $AK = EF$. Find the measure of the angle EKF.



Solution: Since $\angle AGF = 90°$ and $\angle ADF = 90°$, it follows that $ADFG$ is a cyclic quadrilateral. Hence $\angle GAF = \angle GDF = 45°$. Thus the triangle AGF is isosceles and $AG = GF$. In the right angled triangles AGK and FGE, we have $AK = EF$ and $AG = GF$. Hence they are congruent and $GE = KG$. Consequently, the right angled triangle KGE is isosceles and $\angle EKG = 45°$. Thus

$$\angle EKF = 180° - \angle EKG = 135°$$

6.

Let

$$N = 2^5 + 2^{5^2} + 2^{5^3} + \cdots + 2^{5^{2015}}$$

Written in the usual decimal form, find the last two digits of N.

Solution Since $5^n - 5$ is a multiple of 5 and also since $5^n = 1 \mod 4$, it follows that $5^n - 5$ is also divisible by 4. Thus $5^n - 5$ is a multiple of 20 for all n. Also

$$\begin{aligned} 2^{20k} - 1 &= 4^{10k} - 1 \\ &= (1-5)^{10k} - 1 \\ &= 1 - 10k \cdot 5 + \text{ multiples of } 25 \ - 1 \end{aligned}$$

and hence is a multiple of 25. Now for any n, we have

$$\begin{aligned} 2^{5^n} - 2^5 &= 2^5(2^{5^n-5} - 1) \\ &= 2^5(2^{20k} - 1) \\ &= 0 \mod 100 \end{aligned}$$

Thus $2^{5^n} = 2^5 \mod 100$ for all n. Now,

$$\begin{aligned} N &= 2^5 + 2^{5^2} + 2^{5^3} + \cdots + 2^{5^{2015}} \\ &= \underbrace{2^5 + 2^5 + \cdots + 2^5}_{2015 \text{ terms}} \mod 100 \\ &= 2015 \times 32 \mod 100 \\ &= 80 \mod 100 \end{aligned}$$

Thus N ends with 80.